Exercise 2

Evaluate the line integral, where C is the given curve.

$$\int_C (x/y)\,ds, \quad C:\ x = t^3,\ y = t^4,\ 1 \le t \le 2$$

Solution

With the given parameterization in t, the line integral becomes

$$\begin{aligned}\int_C (x/y)\,ds &= \int_1^2 \frac{x(t)}{y(t)}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \int_1^2 \frac{t^3}{t^4}\sqrt{(3t^2)^2 + (4t^3)^2}\,dt \\ &= \int_1^2 \frac{1}{t}\sqrt{9t^4 + 16t^6}\,dt \\ &= \int_1^2 t\sqrt{9 + 16t^2}\,dt.\end{aligned}$$

Make the following substitution.

$$\begin{aligned}u &= 9 + 16t^2 \\ du = 32t\,dt \quad &\rightarrow \quad \frac{du}{32} = t\,dt\end{aligned}$$

Therefore,

$$\begin{aligned}\int_C (x/y)\,ds &= \int_{9+16(1)^2}^{9+16(2)^2} \sqrt{u}\left(\frac{du}{32}\right) \\ &= \frac{1}{32}\int_{25}^{73} u^{1/2}\,du \\ &= \frac{1}{32}\left(\frac{2}{3}u^{3/2}\right)\Big|_{25}^{73} \\ &= \frac{1}{48}\left(73^{3/2} - 25^{3/2}\right) \\ &= \frac{1}{48}\left(73^{3/2} - 125\right).\end{aligned}$$